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Identities and characters for finite groups

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Abstract. Polynomial identities satisfied by certain matrices a, with entries from the group algebra of a finite group G, are derived in the irreducible representations of G. In the special case of the symmetric group S_n the identities obtained parallel those previously encountered by Green for the infinitesimal generators of the general linear group. Moreover, it is demonstrated that the diagonal entries of arbitrary polynomials in the matrix a are central elements whose eigenvalues, in irreducible representations of the group G, are simply determined in terms of the dimensions of the irreducible representations.

1. Introduction

Polynomial identities satisfied by the infinitesimal generators of a semi-simple Lie group have recently been derived (Bracken and Green 1971, Gould 1985, Kostant 1975, O'Brien *et al* 1977) and applied to yield useful information concerning the representation theory of the group. In particular, such identities are useful for the explicit determination of Wigner coefficients (Gould 1987a) and for the evaluation of centraliser elements (Gould 1987a,b) for treating Lie subalgebra embeddings $L_0 \supseteq L$.

In this paper we derive polynomial identities for an arbitrary finite group in the framework of induced representations. Our motivating example is the symmetric group S_n where we have the $(n+1)\times(n+1)$ matrix defined by

$$a_{ij} = \begin{cases} 0 & i=j \\ 1 & i \text{ or } j = n+1 \qquad (i \neq j) \\ (ij) & \text{otherwise} \end{cases} \tag{1}$$

where $(ij)=(ji)$, $i\neq j$, as usual refers to an elementary transposition. Polynomials in the matrix a may be defined recursively according to

$$a_{ij}^{m+1} = \sum_{k=1}^{n+1} a_{ik} a_{kj}^{m} = \sum_{k=1}^{n+1} a_{ik}^{m} a_{kj} \tag{2}$$

where we define

$$a_{ij}^{0} = \delta_{ij}.$$

Our main result is as follows.

Theorem 1.1. Acting on the irreducible representation of S_n with Young diagram

$\lambda=(\lambda_1, \lambda_2, \ldots, \lambda_m)$, the matrix a satisfies the polynomial identity

$$\prod_{r=1}^{m+1}(a-\lambda_r+r-1)=0$$

where we define $\lambda_{m+1}=0$.

The polynomial identities of theorem 1.1 may be regarded as a symmetric group analogue of the polynomial identities obtained by Green (1971) for the infinitesimal generators of the Lie group GL(n). It shall be demonstrated in § 3 that these polynomial identities may, in general, be considerably reduced. As in the case of Green (1971), traces of the matrix powers (2) determine central elements whose eigenvalues, in irreducible representations, are easily determined. We have the following result (notation as above).

Theorem 1.2. Set $\sigma_k=a^k_{n+1\,n+1}$. Then

(i) $\sigma_k=a^k_{ii}$, $i=1, \ldots, n$,

(ii) σ_k belongs to the centre of the group algebra of S_n,

(iii) the eigenvalue $\langle\sigma_k\rangle_\lambda$ of σ_k in the irreducible representation of S_n with Young diagram $\lambda=(\lambda_1, \ldots, \lambda_m)$ is given by

$$\langle\sigma_k\rangle_\lambda=\sum_{r=1}^{m+1}\frac{\alpha_r^k}{(\alpha_r-\alpha_{m+1}+1)}\prod_{l\neq r}^{m+1}\left(\frac{\alpha_r-\alpha_l+1}{\alpha_r-\alpha_l}\right)$$

where $\alpha_r=\lambda_r+1-r$.

The proof of theorem 1.1 is presented in § 3 and the proof of theorem 1.2 is given in § 4. A suitable generalisation of these results for an arbitrary finite group is also obtained.

2. Preliminaries

We shall be concerned with subgroup embeddings $H\supset G$ where G is a subgroup of a finite group H of order $|H|$. We let A(H) (resp A(G)) denote the group algebra of H (resp G) over the complex field $\mathbb{C}$ and we let $Z(H)$ (resp $Z(G)$) denote the centre of A(H) (resp A(G)). Throughout we let $V(\lambda_0)$ (resp $V(\lambda)$) denote an irreducible H (resp G)-module with representation label λ_0 (resp λ). We let π_{λ_0} (resp π_λ) denote the representation afforded by $V(\lambda_0)$ (resp $V(\lambda)$) and set $D[\lambda]=\dim V(\lambda)$. Finally we let $V_H(\lambda)$ denote the H-module induced by the irreducible G-module $V(\lambda)$ and we denote by $\{\lambda\}$ the set of H-representation labels which occur in $V_H(\lambda)$. With this convention we have the H-module decomposition

$$V_H(\lambda)=\bigoplus_{\lambda_0\in\{\lambda\}} m_{\lambda_0}(\lambda)\,V(\lambda_0) \tag{3}$$

where $m_{\lambda_0}(\lambda)$ is the multiplicity of $V(\lambda_0)$ in $V_H(\lambda)$: by the reciprocity theorem of Frobenius (Coleman 1965, Curtis and Reiner 1962) $m_{\lambda_0}(\lambda)$ necessarily equals the multiplicity of $V(\lambda)$ in $V(\lambda_0)$.

Now let $g_1G, g_2G, \ldots, g_mG$ ($m = |H|/|G|$) denote the distinct left G-cosets of H. We let ρ denote the projection $\rho: A(H) \to A(G)$ defined by

$$\rho(h) = \begin{cases} h & h \in G \\ 0 & h \in H \sim G \end{cases}$$

which we extend linearly to all of A(H). The following properties of the mapping ρ are easily established (see, e.g., Coleman 1965, Boerner 1970):

$$\rho(gh) = g\rho(h) \qquad g \in G,\ h \in H \tag{4}$$

$$\rho(g_i^{-1}g_j) = \delta_{ij} \tag{5a}$$

$$\rho(g_i^{-1}hgg_j) = \sum_{k=1}^{m} \rho(g_i^{-1}hg_k)\rho(g_k^{-1}gg_j) \qquad g, h \in H. \tag{5b}$$

We note that equation (4) implies that ρ determines a G-module homomorphism.

We let $gl(m, \mathbb{C})$ denote the space of $m \times m$ matrices over $\mathbb{C}$ and denote by E_{ij} the elementary matrix with 1 in the (i, j) position and zeros elsewhere. We shall be concerned with the map

$$\partial: A(H) \to gl(m, \mathbb{C}) \otimes A(G)$$

defined by

$$\partial(h) = \sum_{i,j=1}^{m} E_{ij} \otimes \rho(g_i^{-1}hg_j) \qquad h \in H$$

which we extend linearly to all of A(H). In terms of the mapping ∂, equations (5a) and (5b) may be expressed

$$\partial(1) = I \otimes 1$$

$$\partial(hg) = \partial(h)\partial(g) \qquad g, h \in H$$

where 1 is the identity element of G and I is the $m \times m$ identity matrix. It follows, in particular, that the mapping ∂ determines an algebra homomorphism.

We may regard elements of the algebra

$$Y = gl(m, \mathbb{C}) \otimes A(G)$$

as $m \times m$ matrices with entries from A(G). In particular if $h \in H$ then $\partial(h)$ is the $m \times m$ matrix with entries

$$\partial(h)_{ij} = \rho(g_i^{-1}hg_j).$$

In matrix notation equation (5) may be expressed

$$\partial(1)_{ij} = \delta_{ij}$$

$$\partial(hg)_{ij} = \sum_{k=1}^{m} \partial(h)_{ik}\partial(g)_{kj} \qquad g, h \in H.$$

3. Polynomial identities

Throughout we let ε denote a fixed (but arbitrary) element of $Z(H)$ and denote the eigenvalue of ε on the irreducible H-module $V(\lambda_0)$ by $\langle\varepsilon\rangle_{\lambda_0}$. We shall be concerned with the $m \times m$ matrix over A(G) defined by

$$a = \partial(\varepsilon) - I \otimes \rho(\varepsilon). \tag{6}$$

We note that $\rho(\varepsilon)$ belongs to the centre of A(G). In particular if C is a conjugacy class of H then we have the central element

$$\varepsilon = \sum_{h \in C} h.$$

In this case

$$\rho(\varepsilon) = \sum_{g \in C \cap G} g$$

and the matrix of equation (6) reduces to

$$a_{ij} = \sum_{h \in C} \rho(g_i^{-1} h g_j) - \delta_{ij} \rho(\varepsilon).$$

In general we have the following result.

Theorem 3.1. Acting on the irreducible G-module $V(\lambda)$ the matrix a satisfies the polynomial identity

$$\prod_{\lambda_0 \in \{\lambda\}} (a - \alpha_{\lambda_0,\lambda}) = 0$$

where

$$\alpha_{\lambda_0,\lambda} = \langle \varepsilon \rangle_{\lambda_0} - \langle \rho(\varepsilon) \rangle_\lambda .$$

Proof. By our construction, acting on $V(\lambda)$ the matrix a may be interpreted as an operator on the induced H-module

$$V_{\mathrm{H}}(\lambda) \simeq \bigoplus_{i=1}^{m} g_i \otimes V(\lambda).$$

We have

$$a \equiv \pi_\lambda^{\mathrm{H}}(\varepsilon) - I \otimes \pi_\lambda[\rho(\varepsilon)]$$

where π_λ^{H} is the representation of H afforded by $V_{\mathrm{H}}(\lambda)$. In view of the decomposition (3) it follows that a takes the constant value

$$\alpha_{\lambda,\lambda_0} = \langle \varepsilon \rangle_{\lambda_0} - \langle \rho(\varepsilon) \rangle_\lambda$$

on each of the irreducible H-modules $V(\lambda_0)$ occurring in $V_{\mathrm{H}}(\lambda)$. The theorem is then seen to follow from the fact that a diagonal matrix d with distinct eigenvalues $\delta_1, \delta_2, \ldots, \delta_k$ satisfies the polynomial identity

$$\prod_{i=1}^{k} (d - \delta_i) = 0.$$

A particular case of the matrix (6) is afforded by the S_n matrix of equation (1). In this case we regard S_n as a subgroup of S_{n+1} and consider the central element

$$\varepsilon = \sum_{i<j}^{n+1} (ij).$$

We clearly have for the case at hand

$$\rho(\varepsilon) = \sum_{i<j}^{n} (ij).$$

As a set of S_n-coset representatives we choose the transpositions $g_i = (in+1)$ ($i = 1, \ldots, n$) together with the identity element $g_{n+1} = 1$. We then have, for $i, j = 1, \ldots, n$,

$$\partial(\varepsilon)_{ij} = \sum_{k<r}^{n} \rho[(in+1)(kr)(jn+1)] + \sum_{k=1}^{n} \rho[(in+1)(kn+1)(jn+1)].$$

Using

$$\rho[(in+1)(kr)(jn+1)] = \begin{cases} \delta_{ij}(kr) & k, r \neq j \\ \delta_{ik}(ij) & r = j \end{cases}$$

$$\rho[(in+1)(kn+1)(jn+1)] = \begin{cases} \delta_{ij}(ki) & k \neq j \\ 0 & \text{otherwise} \end{cases}$$

we obtain

$$\partial(\varepsilon)_{ij} = \begin{cases} (ij) & i \neq j \\ \rho(\varepsilon) & i = j. \end{cases}$$

For the remaining entries we have

$$\partial(\varepsilon)_{in+1} = \partial(\varepsilon)_{n+1i} = \sum_{k=1}^{n} \rho[(in+1)(kn+1)]$$

$$= \sum_{k=1}^{n} \delta_{ik} = 1 \qquad i = 1, \ldots, n$$

$$\partial(\varepsilon)_{n+1n+1} = \sum_{k<r}^{n+1} \rho[(kr)] = \rho(\varepsilon).$$

It follows that the matrix

$$a_{ij} = \partial(\varepsilon)_{ij} - \delta_{ij}\rho(\varepsilon)$$

corresponds to the matrix of equation (1) as required.

We are now in a position to prove theorem 1.1. Let $V(\lambda)$ be an irreducible S_n-module with Young diagram $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ (i.e. λ_r boxes in the rth row). Then the S_{n+1}-module $V_{S_{n+1}}(\lambda)$ induced by $V(\lambda)$ decomposes into irreducible S_{n+1}-modules according to (Hamermesh 1962, Robinson 1961)

$$V_{S_{n+1}}(\lambda) = \bigoplus_{r=1}^{m+1}{}' \, V(\lambda + \Delta_r) \tag{7}$$

where $V(\lambda + \Delta_r)$ denotes the irreducible S_{n+1}-module whose Young diagram is obtained from λ by adding a box to the rth row. The prime in equation (7) signifies that only those S_{n+1}-modules giving rise to allowable Young diagrams are to be retained. From the well known result (Hamermesh 1962)

$$\langle \rho(\varepsilon) \rangle_\lambda = \tfrac{1}{2} \sum_{i=1}^{m} \lambda_i(\lambda_i + 1 - 2i)$$

it follows that on each irreducible S_{n+1}-module $V(\lambda + \Delta_r)$ occurring in equation (7) the matrix a takes the constant value

$$\lambda_r + 1 - r = \langle \varepsilon \rangle_{\lambda+\Delta_r} - \langle \rho(\varepsilon) \rangle_\lambda .$$

The result is then seen to follow immediately from theorem 3.1.

Remarks. The matrix of equation (1) is a particular case of a whole series of tensor matrices which may be defined for S_n in the framework of outer direct products (Coleman 1965, Hamermesh 1962). It shall be demonstrated in a forthcoming publication that a suitable generalisation of theorem 1.1 may be obtained for these higher-order tensor matrices.

We note that the polynomial identity of theorem 1.1 is not, in general, the minimum polynomial identity satisfied by the matrix a. From the proof of theorem 1.1 it follows that, for a given Young diagram λ, only those factors $(a-\lambda_r-r+1)$ corresponding to allowable Young diagrams $(\lambda+\Delta_r)$ need be retained in the identity. Keeping in mind the fact that the roots of the identity of theorem 1.1 are all distinct, we obtain, by this means, the following result (notation as above).

Theorem 3.2. Acting on the irreducible representation of S_n with Young diagram $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_m)$ the matrix a of equation (1) satisfies the minimum polynomial identity

$$(a-\lambda_1)\prod_{\substack{r=2\\ \lambda_r\neq\lambda_{r-1}}}^{m+1}(a-\lambda_r+r-1)=0.$$

The minimum polynomial identities corresponding to some special Young diagrams λ of interest are listed below:

$$\lambda=(1^n) \qquad (a-1)(a+n)=0$$

$$\lambda=(n) \qquad (a-n)(a+1)=0$$

$$\lambda=(pn-p) \qquad (a-p)(a-n+p+1)(a+2)=0$$

$$\lambda=(2^p1^{n-p}) \qquad (a+p-1)(a+n)(a-2)=0.$$

4. Group characters and central elements

Let C denote the centraliser of $\partial[\mathrm{A(H)}]$ in $\mathrm{Y}=\mathrm{gl}(m,\mathbb{C})\otimes\mathrm{A(G)}$, i.e.

$$C=\{y\in\mathrm{Y}|\partial(h)y=y\partial(h),\forall h\in\mathrm{H}\}.$$

We have the following result (notation as in § 2).

Lemma 4.1. Let $\omega\in C$ be arbitrary. Then the diagonal entries of ω are all equal and belong to the centre of A(G), i.e.

$$\omega_{11}=\omega_{22}=\ldots=\omega_{mm}\in\mathrm{Z(G)}.$$

Proof. Suppose $\omega\in C$ and choose $h\in\mathrm{H}$ arbitrary. We then have

$$\partial(h)\omega=\omega\partial(h)$$

which is equivalent to

$$\sum_{k=1}^{m}\rho(g_i^{-1}hg_k)\omega_{kj}=\sum_{k=1}^{m}\omega_{ik}\rho(g_k^{-1}hg_j). \tag{8}$$

Now choose $g \in \mathrm{G}$ arbitrary and set $h = g_i g g_j^{-1}$ in equation (8) to give

$$\sum_{k=1}^{m} \rho(g g_j^{-1} g_k)\omega_{kj} = \sum_{k=1}^{m} \omega_{ik}\rho(g_k^{-1} g_i g)$$

or

$$g \sum_{k=1}^{m} \rho(g_j^{-1} g_k)\omega_{kj} = \sum_{k=1}^{m} \omega_{ik}\rho(g_k^{-1} g_i) g$$

which is equivalent to

$$g\omega_{jj} = \omega_{ii} g \tag{9}$$

where we have used equations (4) and (5a), respectively. Since $g \in \mathrm{G}$ was chosen arbitrarily, equation (9) is seen to hold for all $g \in \mathrm{G}$, $i, j = 1, \ldots, m$, which is sufficient to prove the result.

The above result has important consequences with regard to the characters of the group G. This follows because the matrix a of equation (6) and all of its powers belong to C whence we have the central elements

$$\sigma_k = a_{ii}^k \qquad (i = 1, \ldots, m). \tag{10}$$

We have the following result (notation as in § 2).

Theorem 4.1. The eigenvalue of the invariant σ_k on the irreducible G-module $V(\lambda)$ is given by

$$\langle \sigma_k \rangle_\lambda = \frac{|\mathrm{G}|}{|\mathrm{H}|} \sum_{\lambda_0 \in \{\lambda\}} m_{\lambda_0}(\lambda) \alpha_{\lambda,\lambda_0}^k \frac{D[\lambda_0]}{D[\lambda]}.$$

Proof. Following the derivation of theorem 3.1, acting on the irreducible G-module $V(\lambda)$ the operator a may be regarded as an operator on the induced H-module $V_{\mathrm{H}}(\lambda)$:

$$a \equiv \pi_\lambda^{\mathrm{H}}(\varepsilon) - I \otimes \pi_\lambda[\rho(\varepsilon)].$$

The operator a takes the constant value

$$\alpha_{\lambda_0,\lambda} = \langle \varepsilon \rangle_{\lambda_0} - \langle \rho(\varepsilon) \rangle_\lambda$$

on each irreducible H-module $V(\lambda_0)$ occurring in $V_{\mathrm{H}}(\lambda)$. It follows immediately from the decomposition (3) that the total trace of the matrix power a^k on the space $V_{\mathrm{H}}(\lambda)$ is given by

$$\mathrm{Tr}_\lambda[a^k] = \sum_{\lambda_0 \in \{\lambda\}} m_{\lambda_0}(\lambda) \alpha_{\lambda,\lambda_0}^k D[\lambda_0]. \tag{11}$$

On the other hand we may write, in view of lemma 4.1,

$$\begin{aligned} \mathrm{Tr}_\lambda[a^k] &= \sum_{i=1}^{m} \mathrm{Tr}\ \pi_\lambda(a_{ii}^k) \\ &= m\ \mathrm{Tr}\ \pi_\lambda(\sigma_k) \\ &= \frac{|\mathrm{H}|}{|\mathrm{G}|} D[\lambda] \langle \sigma_k \rangle_\lambda. \end{aligned}$$

The result is then seen to follow by comparison with equation (11).

In the case of the symmetric group S_n we have the central elements

$$\sigma_k = a^k_{n+1\,n+1} \tag{12}$$

where a is the matrix of equation (1). Thus parts (i) and (ii) of theorem 1.2 follow immediately from lemma 4.1. The k-cycle class invariants (Coleman 1965, Robinson 1961)

$$c_1 = 1 \qquad c_k = \frac{1}{k} \sum_{i_1 \neq i_2 \neq \ldots \neq i_k} (i_1 i_2 \ldots i_k) \qquad k = 2, \ldots, n$$

are related to the invariants (12) by the following lemma.

Lemma 4.2.

$$\sigma_{k+1} = kc_k + \phi \qquad \text{where } \phi \in \mathbb{C}[c_1, c_2, \ldots, c_{k-1}]$$

$(k = 2, \ldots, n)$ with $\sigma_2 = n$. In particular, the invariants $\sigma_2, \ldots, \sigma_{n+1}$ generate (as an algebra) the centre of the symmetric group algebra.

Proof. By definition it follows that $\sigma_2 = n$ and for $k = 2, \ldots, n$ we have

$$\begin{aligned}
\sigma_{k+1} &= \sum_{i_1, i_2, \ldots, i_k}^{n+1} a_{n+1\,i_1} a_{i_1 i_2} \ldots a_{i_{k-1} i_k} a_{i_k\, n+1} \\
&= \sum_{i_1 \neq i_2 \neq \ldots \neq i_k} (i_1 i_2)(i_2 i_3) \ldots (i_{k-1} i_k) + \phi \\
&= kc_k + \phi
\end{aligned}$$

where ϕ is the sum of all terms of the form

$$a_{i_1 i_2} a_{i_2} a_{i_3} \ldots a_{i_{k-1} i_k}$$

with at least two indices i_r, i_j $(r \neq j)$ equal or with at least one index i_r equal to $n+1$. This is enough to ensure that ϕ is a sum of products of at most $(k-2)$ transpositions whence ϕ belongs to $\mathbb{C}[c_1, c_2, \ldots, c_{k-1}]$. Since the c_k $(k = 1, \ldots, n)$ generate the centre of the group algebra of S_n (see, e.g., Coleman 1965, Robinson 1961) the same must be true of the invariants $\sigma_2, \sigma_3, \ldots, \sigma_{n+1}$.

To complete the proof of theorem 1.2, we conclude by establishing part (iii) of theorem 1.2. We have from theorem 4.1 that the eigenvalues of the S_n invariants (12) in the irreducible representation with Young diagram $\lambda = (\lambda_1, \ldots, \lambda_m)$ are given by (notation as in § 1)

$$\langle \sigma_k \rangle_\lambda = \frac{1}{n+1} \sum_{r=1}^{m+1}{}' \alpha_r^k \frac{D[\lambda + \Delta_r]}{D[\lambda]} \tag{13}$$

where the prime indicates that the summation is over all r such that $(\lambda + \Delta_r)$ is an allowable Young diagram for S_{n+1}. For such r we obtain, from the hook-length formula of Robinson (1961),

$$\frac{D[\lambda + \Delta_r]}{D[\lambda]} = \left(\frac{n+1}{\beta_r + 1}\right) \prod_{l \neq r}^{m} \left(\frac{\beta_r - \beta_l + 1}{\beta_r - \beta_l}\right) \qquad r = 1, \ldots, m$$

where $\beta_r = \lambda_r + m - r (r = 1, \ldots, m)$. In the case $r = m+1$ we obtain

$$\frac{D[\lambda + \Delta_{m+1}]}{D[\lambda]} = (n+1) \prod_{r=1}^{m} \left(\frac{\beta_r}{\beta_r + 1}\right).$$

In terms of the characteristic roots $\alpha_r = \lambda_r + 1 - r$ the above formulae may be written in the unified form

$$\frac{D[\lambda+\Delta_r]}{D[\lambda]} = \frac{n+1}{(\alpha_r - \alpha_{m+1} + 1)} \prod_{l\neq r}^{m+1} \left(\frac{\alpha_r - \alpha_l + 1}{\alpha_r - \alpha_l}\right) \qquad r = 1, \ldots, m+1.$$

Substituting into equation (13) we obtain

$$\langle \sigma_k \rangle_\lambda = \sum_{r=1}^{m+1}{}' \frac{\alpha_r^k}{(\alpha_r - \alpha_{m+1} + 1)} \prod_{l\neq r}^{m+1} \left(\frac{\alpha_r - \alpha_l + 1}{\alpha_r - \alpha_l}\right). \tag{14}$$

If the Young diagram $(\lambda + \Delta_r)$ is not allowable we must have $\lambda_r = \lambda_{r-1}$ in which case $\alpha_r - \alpha_{r-1} + 1 = 0$. Since the product

$$\prod_{l\neq r}^{m+1} \left(\frac{\alpha_r - \alpha_l + 1}{\alpha_r - \alpha_l}\right)$$

always contains a factor $(\alpha_r - \alpha_{r-1} + 1)$ we see in fact that the summation in equation (14) may be extended over all $r = 1, \ldots, m+1$. Finally, as previously noted, the characteristic roots α_r $(r = 1, \ldots, m+1)$ are all distinct so that the products in equation (14) are well defined. This completes the proof of the theorem.

Remarks. Our approach to theorem 4.1 was motivated by the Casimir invariant formulae for the simple Lie algebras derived by Edwards (1978) and O'Kubo (1977) (see also Gould 1987b). In the case of the symmetric group S_n it has been demonstrated (Hamermesh 1962, Robinson 1961) that eigenvalue formulae of the type in theorem 1.2 are useful for the calculation of group characters. It is hoped that the general result of theorem 4.1 may be of similar use for a wider range of groups. It would be of particular interest to examine in further detail the case where G is a simple group. From the point of view of obtaining central elements and their eigenvalues it suffices to make the simplest choice for the matrix a, i.e. the simplest (non-trivial) choice for $\varepsilon \in Z(\mathrm{H})$ and the group $\mathrm{H} \neq \mathrm{G}$ in which to embed G.

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